AD-A088 242 BROWN UNIV PROVIDENCE RI LEFSCHETZ CENTER FOR DYNAMI--ETC F/G 12/1
GENERIC PROPERTIES OF AN INTEGRO-DIFFERENTIAL EQUATION.(U)
JUN 80 J K HALE DAAG29-79-C-0161
UNCLASSIFIED AFOSR-TR-80-0575 NL

1 OF 1

END
DATE
FILMED
9-80
DTIC

unclassified

REPORT DOCUMENTATION PAGE — LEVEL II

READ INSTRUCTIONS BEFORE COMPLETING FORM

(3)

1. REPORT NUMBER: (19) AFOSR-TR- 80-0575 (17)
2. GOVT ACCESSION NO.: AD-A088 242
3. RECIPIENT'S CATALOG NUMBER:

4. TITLE (and Subtitle): (6) GENERIC PROPERTIES OF AN INTEGRO-DIFFERENTIAL EQUATION,

5. TYPE OF REPORT & PERIOD COVERED: (9) Interim rept.

6. PERFORMING ORG. REPORT NUMBER:

7. AUTHOR(s): (10) JACK K./HALE (12) 24

8. CONTRACT OR GRANT NUMBER(s): AFOSR 76-3092

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
DIVISION OF APPLIED MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RHODE ISLAND 02912

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F (16) 2304 (17) A1

11. CONTROLLING OFFICE NAME AND ADDRESS:
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH/NM
BOLLING AIR FORCE BASE
WASHINGTON, D.C.

12. REPORT DATE: June, 1980 (11) Jun 80

13. NUMBER OF PAGES: 11

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): (15) DAAG-29-79-C-0161, AFOSR-76-3092

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DTIC ELECTE AUG 26 1980 S D E

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Consider the functional differential equation where a,g are continuous. The linear function is such that the characteristic equation has two eigenvalues on the imaginary axis and the remaining ones with negative real parts. In spite of this, it is shown there is no generic Hopf bifurcation for any g. The nature of the bifurcation is characterized under hypotheses which appear to be generic in g.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 401834 alt

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AFOSR-TR- 80-0575

GENERIC PROPERTIES OF AN INTEGRO-DIFFERENTIAL EQUATION

Accession For	
NTIS GRA&I	☑
DDC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist.	Avail and/or special
A	

by

Jack K. Hale+

412 Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

June, 1980

+This research was supported in part by the National Science Foundation, under MCS 79-05774, in part by the U.S. Army under AROD-DAAG 29-79-C-0161, and in part by the Air Force Office of Scientific Research under AFOSR 76-3092C.

80 8 14 116

GENERIC PROPERTIES OF AN INTEGRO-DIFFERENTIAL EQUATION

Jack K. Hale

Abstract: Consider the functional differential equation

$$\dot{x}(t) = -\int_{-1}^{0} a(-\theta)g(x(t+\theta))d\theta$$

where a,g are continuous, $a \geq 0$, $a(1) = 0$, $g(0) = 0$, $g'(0) = 1$, $xg(x) > 0$ for $x \neq 0$. The linear function $a_0(s) = 4\pi^2(1-s)$ is such that the characteristic equation

$$\lambda + \int_{-1}^{0} a_0(-\theta)e^{\lambda\theta}d\theta = 0$$

has two eigenvalues on the imagainary axis and the remaining ones with negative real parts. In spite of this, it is shown there is no generic Hopf bifurcation for any g. The nature of the bifurcation is characterized under hypotheses which appear to be generic in g.

1. Introduction. Consider the functional differential equation

$$\dot{x}(t) = -\int_{-1}^{0} a(-\theta)g(x(t+\theta))d\theta \tag{1.1}$$

where $a \in C([0,1],\mathbb{R})$, $g \in C^3(\mathbb{R},\mathbb{R})$, $a \geq 0$, $a(1) = 0$, $g(0) = 0$, $g'(0) = 1$. The characteristic equation

$$\lambda = -\int_{-1}^{0} a(-\theta)e^{\lambda\theta}d\theta \tag{1.2}$$

for the linear variational equation around zero for $a = a_0$, $a_0(s) = 4\pi^2(1-s)$, has all solutions with real parts < 0 except two on the imaginary axis given by $\pm 2\pi i$. Furthermore, there is a neighborhood U of a_0 and a submanifold Γ of codimension one in $C([0,1],\mathbb{R})$ such that $U \setminus \Gamma = U_1 \cup U_2$ with $U_1 \cap U_2$ the empty set, for $a \in U_1$, all solutions of (1.2) have negative real parts and, for $a \in U_2$, all have negative real parts except two which have positive real parts.

Under such a circumstance, one expects to obtain a generic Hopf bifurcation at a_0 for a residual set of $g \in C^3(\mathbb{R},\mathbb{R})$. That is, for a residual set of g, a unique hyperbolic periodic orbit should bifurcate from zero as a crosses Γ from U_1 (respectively U_2) to U_2 (respectively U_1). We show this is not the case. Under an assumption which appears to be generic in $g \in C^5(\mathbb{R},\mathbb{R})$, we characterize the nature of the bifurcation point a_0.

This result shows that the generic properties of Equation (1.1) with the integrand restricted to a product of two functions $a(s)g(x)$ is completely different from the generic properties that

would be obtained by considering general functions $h(s,x)$ of two variables.

For notational purposes, we let $C = C([-1,0],\mathbb{R})$. For any $\phi \in C$, we suppose the solution $x(\phi)(t)$ through ϕ is defined for $t \geq 0$. Let $T_{a,g}(t): C \to C$, $t \geq 0$, be the semigroup operator by (1.1); that is, $T_{a,g}(t)\phi(\theta) = x(\phi)(t+\theta)$, $-1 \leq \theta \leq 0$.

2. Nongeneric Hopf bifurcation. To prove there can be no generic Hopf bifurcation for Equation (1.1), we need the following proposition from Hale [3, p. 122] or Levin and Nohel [5].

Proposition 2.1. If $\dot{a} \leq 0$, $\ddot{a} \geq 0$, $xg(x) > 0$ for all x, $G(x) = \int_0^x g \to \infty$ as $|x| \to \infty$, then every solution of (1.1) is bounded and

(i) if there is an s such that $\ddot{a}(s) > 0$, then every solution approaches zero as $t \to \infty$;

(ii) if $\ddot{a}(s) = 0$ for all s (that is, a is linear) then, for any $\phi \in C$, there is either an equilibrium point or a one-periodic solution $p = p(\phi)$ of the ordinary differential equation

$$\ddot{y} + a(0)g(y) = 0 \tag{2.1}$$

such that the ω-limit set of the orbit through ϕ is $\{p_t,\ t \in \mathbb{R}\}$, where $p_t \in C$, $p_t(\theta) = p(t+\theta)$, $-1 \leq \theta \leq 0$.

Since $g'(0) = 1$ in (1.1), we may choose a neighborhood of $x = 0$ and extend g outside this neighborhood so that it satisfies

the condition of Proposition 2.1 and so that Equation (2.1) has no one-periodic solutions outside U for any a in a sufficiently small neighborhood V of $a_0(s) = 4\pi^2(1-s)$. From the results in Cooperman [1] (see also Hale [14]), the semigroup $T_{a,g}(t)$ has a maximal compact invariant set $A_{a,g}$ which belongs to the set $\tilde{U} = \{\phi \in C: \phi(\theta) \in U, -1 \leq \theta \leq 0\}$ for each $a \in V$. Also, this set is uniformly asymptotically stable and attracts bounded sets of C. The set $A_{a,g}$ is upper semicontinuous in (a,g).

Proposition 2.2. *For any real* μ *such that* $\mu a_0 \in V$, *the set* $A_{\mu a_0,g}$ *cannot contain a uniformly asymptotically stable periodic orbit*.

Proof. Suppose $A_{\mu a_0,g}$ contains a periodic orbit γ which is uniformly asymptotically stable. Then, for any neighborhood U of γ there is a neighborhood V of γ and a neighborhood W of μa_0 in the C^0-topology such that, for any $a \in W$, $\phi \in V$, the positive orbit $\gamma^+(\phi)$ of (1.1) belongs to U. One can choose the neighborhood U of γ and W of μa_0 so that U does not contain the equilibrium point zero of (1.1) for any $a \in W$.

In the neighborhood W of μa_0 there exists a strictly convex function a. Thus, $A_{a,g}$ contains no periodic orbits and every solution of (1.1) approaches zero by Proposition 2.1. This contradicts the fact that some positive orbits remain in U and proves that no periodic orbit can be uniformly asymptotically stable.

Let $\omega(b)$ be the period of the solution of (2.1) through

$(b,0)$, $b > 0$, for (a_0,g). There is an open dense set of g such that $\omega'(b) \neq 0$. Suppose g is chosen so that this is true. Then there is a neighborhood U of $x = 0$ such that Equation (2.1) for (a_0,g) has no one-periodic solution in U. Thus, we may assume g extended so that $A_{a_0,g} = \{0\}$. Suppose there is a generic Hopf bifurcation at (a_0,g). Since $A_{a,g}$ is upper semicontinuous at (a_0,g), $A_{a,g}$ is uniformly asymptotically stable and the bifurcated periodic orbit is hyperbolic, it follows that the periodic orbit must be uniformly asymptotically stable. This contradicts Proposition 2.2. This proves there cannot be a residual set of g for which there is a generic Hopf bifurcation.

3. The bifurcations at a_0. To understand better the nature of the bifurcationsat a_0, we give a more detailed analysis of the equations characterizing the bifurcation of periodic orbits from zero.

We need the following lemma.

Lemma 3.1. *There is a neighborhood* U *of* a_0 *in the* C^0*-topology, a* $\delta > 0$ *and an analytic function* $\lambda^*: U \to \mathbb{C}$ *such that* $\lambda^*(a_0) = 2\pi i$ *and, for every* $a \in U$, *the equation* (1.2) *has exactly one solution in each of the circles* $|\lambda \pm 2\pi i| < \delta$ *given respectively by* $\lambda^*(a), \bar{\lambda}^*(a)$ *and all other solutions with real parts* $\leq -\delta$. *Furthermore, if*

$$\Gamma^- = \{a \in U: \ \mathrm{Re}\ \lambda^*(a) < 0\}$$

$$\Gamma^0 = \{a \in U: \ \mathrm{Re}\ \lambda^*(a) = 0\}$$

$$\Gamma^+ = \{a \in U: \ \mathrm{Re}\ \lambda^*(a) > 0\}$$

then each of these sets is nonempty and Γ^0 is a submanifold of codimension one.

Proof. If

$$F(\lambda, a) = \lambda + \int_{-1}^{0} a(-\theta) e^{\lambda\theta} d\theta$$

then $F(2\pi i, a_0) = 0$ and $\partial f/\partial\lambda = 1 - 8\pi^2$ at $(2\pi i, a_0)$. The Implicit Function Theorem implies the existence of a function $\lambda^*(a) \in \mathbb{C}$ analytic in a neighborhood of a_0 with $\lambda^*(a_0) = 2\pi i$. The other properties of λ^* follow essentially from Rouché's Theorem.

To show Γ^0 has codimension one, consider the family of functions $(a_0+\nu b_0)$, $b_0(s) = 4\pi^2 s(1-s)$, $\nu \in \mathbb{R}$. Then the derivative of $\lambda^*(a_0+\nu b_0)$ with respect to ν at $\nu = 0$ is easily seen

$$\left.\frac{\partial\lambda^*}{\partial\nu}\right|_{\nu=0} = -4\pi^2 \int_{-1}^{0} \theta(1+\theta) e^{2\pi i\theta} d\theta$$

Thus,

$$\left.\frac{\partial \mathrm{Re}\ \lambda^*}{\partial\nu}\right|_{\nu=0} = -4\pi^2 \int_{-1}^{0} \theta(1-\theta) \cos 2\pi\theta\, d\theta > 0.$$

This shows that Γ^0 has codimension one and also that Γ^-, Γ^+ are not empty. This proves the lemma.

Remark 3.2. We knew from the previous section that $\mathrm{Re}\,\lambda^*(a) < 0$ if $a \in U$ is strictly convex. The above proof shows that $\mathrm{Re}\,\lambda^*(a) > 0$ if $a \in U$ is of the form $a = a_0 + b$ where b is strictly concave, $b(0) = b(1) = 0$.

For $a = a_0$, the characteristic equation for the linear variational equation around zero has two purely imaginary roots. For a near a_0, and a neighborhood W of zero let $B(r,a,g)$ be the scalar bifurcation function obtained by applying the usual method of Liapunov-Schmidt for the periodic solutions of (1.1) in W which for $a = a_0$ are equal to $r \cos 2\pi t$ (see, for example, deOliveira and Hale [2]). This function has the property that the periodic solutions of the type specified are in one to one correspondence with the nonnegative zeros of $B(r,a,g)$. Furthermore, the stability properties of the periodic solution corresponding to a zero r_0 of $B(r,a,g)$ when restricted to a center manifold at $x = 0$ are the same as the stability properties of the equilibrium point r_0 of the scalar equation

$$\dot{r} = B(r,a,g) \tag{3.1}$$

(see deOliveira and Hale [2]). The function $B(r,a,g)$ is an odd function of r and has five continuous derivatives if $g \in C^5(\mathbb{R},\mathbb{R})$. Let

$$(3.2) \quad B(r,a,g) = \alpha_1(a,g)r + \alpha_3(a,g)r^3 + \alpha_5(a,g)r^5 + o(|r|^5)$$

as $r \to 0$.

If $\lambda^*(a)$ is the function given in Lemma 3.1, the manner in which the bifurcation function is constructed implies that

$$(3.3) \quad \begin{array}{l} \alpha_1(a,g) = 0 \text{ if and only if } \operatorname{Re} \lambda^*(a) = 0 \\ \operatorname{sign} \alpha_1(a,g) = \operatorname{sign} \operatorname{Re} \lambda^*(a). \end{array}$$

Thus, $\alpha_1(a_0,g) = 0$.

Let $\alpha_3^0 = \alpha_3(a_0,g)$. We now show that $\alpha_3^0 = 0$. Since the solution $x = 0$ of Equation (1.1) for $a = a_0$ is asymptotically stable, it follows that the zero solution of (3.1) for $a = a_0$ is asymptotically stable. Thus, $\alpha_3^0 \le 0$. A generic Hopf bifurcation corresponds to $\alpha_3^0 < 0$. We have shown in the previous section that $\alpha_3^0 = \alpha_3(a_0,g) = 0$ for an open dense set of g's. Thus, $\alpha_3^0 = 0$ for all g since it is continuous in g. This shows there is no generic Hopf bifurcation for any g.

Again, the stability of the zero solution of Equation (1.1) for $a = a_0$ implies that $\alpha_5(a_0,g) \le 0$. We make the hypothesis that

$$(3.4) \quad \alpha_5^0(g) \overset{\text{def}}{=} \alpha_5(a_0,g) < 0.$$

This implies that

$$(3.5) \quad B(r,a_0,g) = \alpha_5^0(g)r^5 + o(r^5), \quad \alpha_5^0 < 0.$$

We have not made the computations (which would be extremely complicated) to obtain the constant $\alpha_5^0(g)$. However, it certainly seems plausible that the set of g for which $\alpha_5^0(g) < 0$ is open in the space $C^5(U,\mathbb{R})$ for a given bounded neighborhood U of $x = 0$.

If $B(r,a,g) = rP(r^2,a,g)$, then

$$(3.6) \qquad P(\rho,a,g) = \alpha_1(a,g) + \alpha_3(a,g)\rho + \alpha_5(a,g)\rho^2 + o(\rho^2)$$

as $\rho \to 0$. This function has a unique maximum $\eta(a,g)$ in a neighborhood U of $a = a_0$ which occurs at a value $\rho^*(a,g)$ and $\eta(a_0,g) = 0$. Let

$$(3.7) \qquad \begin{aligned} SN^0 &= \{a \in U: \eta(a,g) = 0,\ \rho^*(a,g) \geq 0\} \\ SN^{+(-)} &= \{a \in U: \eta(a,g) > (<) 0,\ \rho^*(a,g) \geq 0\}. \end{aligned}$$

One can show that every tangent vector to SN^0 is a tangent vector to Γ^0. We suppose that

$$(3.8) \qquad SN^0 \text{ is a submanifold of codimension 1, } S_{N+} \neq \phi.$$

It is possible to show that hypothesis (3.8) is satisfied for an open set of $g \in C^5(W,\mathbb{R})$ for a bounded neighborhood W of zero.

We can now prove the following result.

Theorem 3.3. With hypotheses (3.4) and (3.8), there is a neighborhood U of a_0 in the C^0-topology and a neighborhood W of $x = 0$ such that U is subdivided into regions as shown in Figure 1, the set $A_{a,g}$ is a disk for each $a \in U$ with boundary being a periodic orbit and the flow on a two dimensional manifold in $A_{a,g}$ is shown in Figure 1.

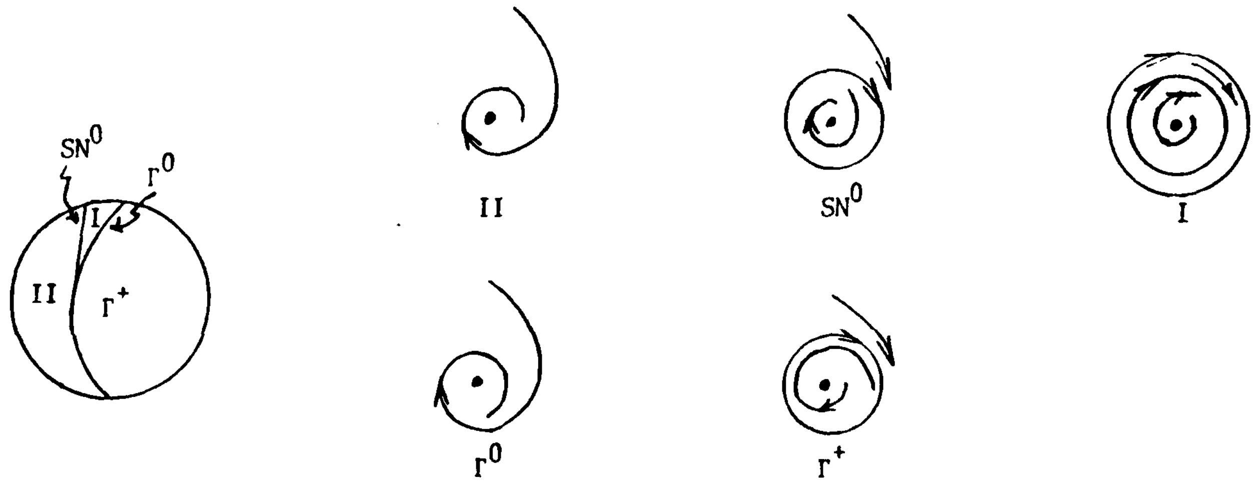

Figure 1.

Proof. If $a \in \Gamma^+$, then $\alpha_1(a,g) > 0, \alpha_3(a,g) < 0$ implies that $P(p,a,g)$ in (3.6) has a unique positive zero. Thus, there is a unique periodic solution in a small neighborhood of zero and it is asymptotically stable as shown in the flow for Γ^+. This shows that $SN \subseteq \Gamma^-$. By hypothesis (3.8) and the fact that the stability properties of the priodic orbits are determined by (3.1), we have that the flow on SN^0 is the one shown in Figure 1. Also, the flow in the other two regions must be one of those shown in Figure 1. We only need to verify that the regions are ordered as shown. In the proof of

Lemma 3.1 we showed that the curve $(a_0+\nu b_0)$, $b_0(s) = 4\pi^2 s(1-s)$ was transversal to Γ^0 at $\nu = 0$. For $\nu < 0$ this function is in Γ^- and stictly convex. Thus, the origin is uniformly asymptotically stable. This proves that the flow in Region II in Figure 1 is the one that is depicted. This proves the theorem.

REFERENCES

[1] Cooperman, G.D., α-condensing maps and dissipative systems. Ph.D. Thesis, Brown University, June 1978.

[2] deOliveira, J.C. and J.K. Hale, Dynamic behavior from bifurcation equations. Tohôku Math. J. to appear.

[3] Hale, J.K., Theory of Functional Differential Equations, Applied Math. Sci., Vol. 3, 2nd Edition, Springer-Verlag, 1977.

[4] Hale, J.K., Some recent results on dissipative processes. Proc. Symposium on Functional Differential Equations and Dynamical Systems, São Carlos, Brazil, July, 1979. To appear in Springer Lecture Notes in Math.

[5] Levin, J.J. and J. Nohel, On a nonlinear delay equation. J. Math. Ana. Appl. 8(1964), 31-44.

DA
FILM
9 -